

On the colored Jones polynomials of ribbon links, boundary links and Brunnian links

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November 27, 2011

Abstract

Habiro gave principal ideals of $\mathbb{Z}[q, q^{-1}]$ in which certain linear combinations of the colored Jones polynomials of algebraically-split links take values. The author proved that the same linear combinations for ribbon links, boundary links and Brunnian links are contained in smaller ideals of $\mathbb{Z}[q, q^{-1}]$ generated by several elements. In this paper, we prove that these ideals also are principal, each generated by a product of cyclotomic polynomials.

1 Introduction

After the discovery of the Jones polynomial, Reshetikhin and Turaev [7] defined an invariant of framed links whose components are colored by finite dimensional representations of a ribbon Hopf algebra. The *colored Jones polynomial* can be defined as the Reshetikhin-Turaev invariant of links whose components are colored by finite dimensional representations of the quantized enveloping algebra $U_h(sl_2)$.

We are interested in the relationship between *algebraic properties* of the colored Jones polynomial and *topological properties* of links.

In this paper, we consider the following three types of links.

A link is called a *ribbon link* if it bounds the image of an immersion from a disjoint union of disks into S^3 with only ribbon singularities.

An n -component link $L = L_1 \cup \cdots \cup L_n$ is called a *boundary link* if it bounds a disjoint union of n Seifert surfaces F_1, \dots, F_n in S^3 such that L_i bounds F_i for $i = 1, \dots, n$.

A link L is called a *Brunnian link* if every proper sublink of L is trivial.

In [4], Habiro used certain linear combinations $J_{L; \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}}, l_1, \dots, l_n \geq 0$, of the colored Jones polynomials of a link L to construct the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. He proved that $J_{L; \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}}$ for an algebraically-split, 0-framed link L is contained in a certain principal ideal of $\mathbb{Z}[q, q^{-1}]$ (Theorem 2.1). This result was improved by the present author [8, 9, 10, 11] in the

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special case of ribbon links, boundary links (Theorem 2.2) and Brunnian links (Theorem 2.4) by using ideals I_{l_1}, \dots, I_{l_n} of $\mathbb{Z}[q, q^{-1}]$, where Theorem 2.2 for boundary links had been conjectured by Habiro [4]. Here, in [8], we gave an alternative proof of the fact that the Jones polynomial of an n -component ribbon link is divisible by the Jones polynomial of the n -component trivial link, which was proved first by Eisermann [1]. The results in [4, 8, 9, 10, 11] are proved by using the *universal sl_2 invariant of bottom tangles* (cf. [3, 4]), which has the universality property for the colored Jones polynomial of links.

In this paper, we prove that the ideal I_l , $l \geq 0$, is a principal ideal generated by a product of cyclotomic polynomials (Theorem 3.1), and rewrite Theorems 2.1, 2.2 and 2.4 by using these generators (Proposition 3.3).

2 Results for the colored Jones polynomial

In this section, we recall results in [4, 9, 10, 11] for the colored Jones polynomial. For the definition of the quantized enveloping algebra $U_h(sl_2)$, see, e.g., [6, 4, 9]. We set $q = \exp h$.

For $m \geq 1$, let V_m denote the m -dimensional irreducible representation of $U_h(sl_2)$. Let \mathcal{R} denote the representation ring of $U_h(sl_2)$ over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., \mathcal{R} is the $\mathbb{Q}(q^{\frac{1}{2}})$ -algebra

$$\mathcal{R} = \text{Span}_{\mathbb{Q}(q^{\frac{1}{2}})} \{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

Habiro [4] studied the following elements in \mathcal{R}

$$\tilde{P}'_l = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}),$$

for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres.

For the definition of the colored Jones polynomial $J_{L; X_1, \dots, X_n}$ of L with i th component L_i colored by $X_i \in \mathcal{R}$, see, e.g., [5, 4, 9].

Set

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q \{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$

for $i \in \mathbb{Z}, n \geq 0$.

Habiro [4] proved the following.

Theorem 2.1 (Habiro [4]). *Let L be an n -component, algebraically-split link with 0-framing. We have*

$$J_{L; \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}} \in \frac{\{2l_{\max} + 1\}_{q, l_{\max}+1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}], \quad (1)$$

for $l_1, \dots, l_n \geq 0$, where $l_{\max} = \max(l_1, \dots, l_n)$.

Set

$$f_{l,k} = \{l-k\}_q! \{k\}_q!,$$

for $0 \leq k \leq l$. For $l \geq 0$, let I_l be the ideal of $\mathbb{Z}[q, q^{-1}]$ generated by $f_{l,0}, \dots, f_{l,l}$.

In [9, 10], we proved the following.

Theorem 2.2 ([9, 10]). *Let L be an n -component ribbon or boundary link with 0-framing. For $l_1, \dots, l_n \geq 0$, we have*

$$J_{L; \tilde{P}'_{l'_1}, \dots, \tilde{P}'_{l'_n}} \in \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q} \prod_{1 \leq i \leq n, i \neq i_M} I_{l_i}, \quad (2)$$

where $l_{\max} = \max(l_1, \dots, l_n)$ and i_M is an integer such that $l_{i_M} = l_{\max}$.

Remark 2.3. Theorem 2.2 for boundary links had been conjectured by Habiro [4].

In [11], we prove the following.

Theorem 2.4 ([11]). *Let L be an n -component Brunnian link with $n \geq 3$. We have*

$$J_{L; \tilde{P}'_{l'_1}, \dots, \tilde{P}'_{l'_n}} \in \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q \{l_{\min}\}_q!} \prod_{1 \leq i \leq n, i \neq i_M, i_m} I_{l_i}, \quad (3)$$

for $l_1, \dots, l_n \geq 0$, where $l_{\max} = \max(l_1, \dots, l_n)$, $l_{\min} = \min(l_1, \dots, l_n)$ and $i_M, i_m, i_M \neq i_m$, are integers such that $l_{i_M} = l_{\max}$, $l_{i_m} = l_{\min}$, respectively.

Let us compare Theorems 2.1, 2.2 and 2.4. For $l_1, \dots, l_n \geq 0$, let $Z_a^{(l_1, \dots, l_n)}$, $Z_{r,b}^{(l_1, \dots, l_n)}$ and $Z_{Br}^{(l_1, \dots, l_n)}$ denote the ideals of $\mathbb{Z}[q, q^{-1}]$ at the right hand sides of (1), (2) and (3), respectively, i.e., we set

$$\begin{aligned} Z_a^{(l_1, \dots, l_n)} &= \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}], \\ Z_{r,b}^{(l_1, \dots, l_n)} &= \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q} \prod_{1 \leq i \leq n, i \neq i_M} I_{l_i}, \\ Z_{Br}^{(l_1, \dots, l_n)} &= \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q \{l_{\min}\}_q!} \prod_{1 \leq i \leq n, i \neq i_M, i_m} I_{l_i}. \end{aligned}$$

For $l_1, \dots, l_n \geq 0$, we have

$$Z_{r,b}^{(l_1, \dots, l_n)} \subset Z_a^{(l_1, \dots, l_n)}, \quad Z_{r,b}^{(l_1, \dots, l_n)} \subset Z_{Br}^{(l_1, \dots, l_n)},$$

since we have

$$\begin{aligned} Z_{r,b}^{(l_1, \dots, l_n)} &= \left(\prod_{1 \leq i \leq n, i \neq i_M} I_{l_i} \right) \cdot Z_a^{(l_1, \dots, l_n)}, \\ &= (\{l_{\min}\}_q! I_{l_{\min}}) \cdot Z_{Br}^{(l_1, \dots, l_n)}. \end{aligned}$$

On the other hand, there are no inclusion which satisfies for all $l_1, \dots, l_n \geq 0$ between $Z_a^{(l_1, \dots, l_n)}$ and $Z_{Br}^{(l_1, \dots, l_n)}$. For example, we have $Z_a^{(2,2,2,2)} \not\subset Z_{Br}^{(2,2,2,2)}$ and $Z_{Br}^{(2,2,2,2)} \not\subset Z_a^{(2,2,2,2)}$ since

$$\begin{aligned} Z_a^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q} \mathbb{Z}[q, q^{-1}] \\ &= (q-1)^2(q+1)(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1) \mathbb{Z}[q, q^{-1}], \\ Z_{Br}^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q \{2\}_{q!}} \{1\}_q^4 \mathbb{Z}[q, q^{-1}] \\ &= (q-1)^4(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1) \mathbb{Z}[q, q^{-1}]. \end{aligned}$$

Since a Brunnian link with $n \geq 3$ components is algebraically-split with 0-framing, we have the following refinement of Theorem 2.4.

Theorem 2.5. *Let L be an n -component Brunnian link with $n \geq 3$. We have*

$$J_{L; \tilde{P}'_1, \dots, \tilde{P}'_n} \in Z_a^{(l_1, \dots, l_n)} \cap Z_{Br}^{(l_1, \dots, l_n)},$$

for $l_1, \dots, l_n \geq 0$.

3 Main result for the ideal I_l

In this section, we state the main result of this paper.

For $l \geq 0$, recall the generators $f_{l,0}, \dots, f_{l,l}$ of the ideal I_l . Set

$$g_l = \text{GCD}(f_{l,0}, \dots, f_{l,l}).$$

It is clear that $I_l \subset g_l \mathbb{Z}[q, q^{-1}]$. The opposite inclusion follows if and only if I_l is principal. Since $\mathbb{Z}[q, q^{-1}]$ is *not* a principal ideal domain, it had been a problem if I_l is principal or not. The main result in this paper (Theorem 3.1) is that I_l is principal, where we determine g_l explicitly. The proof is in Section 4.

For $m \geq 1$, let $\Phi_m = \prod_{d|m} (q^d - 1)^{\mu(\frac{m}{d})} \in \mathbb{Z}[q]$ denote the m th cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all positive divisors d of m , and μ is the Möbius function. For $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to r .

Theorem 3.1. *For $l \geq 0$, the ideal I_l is the principal ideal generated by g_l . Moreover, we have*

$$g_l = \prod_{m \geq 1} \Phi_m^{t_{l,m}}, \tag{4}$$

where

$$t_{l,m} = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 & \text{for } 1 \leq m \leq l, \\ 0 & \text{for } l < m. \end{cases}$$

Here is a table of $t_{l,m}$ for $1 \leq m \leq 4$, $0 \leq l \leq 16$.

$m \setminus l$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7
3	0	0	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4
4	0	0	0	0	0	0	0	1	1	1	1	2	2	2	2	3	3

Remark 3.2. In [11], Theorem 3.1 is used in the proof of Theorem 2.4.

Theorem 3.1 implies that the ideals $Z_{r,b}^{(l_1, \dots, l_n)}$ and $Z_{Br}^{(l_1, \dots, l_n)}$ are principal. Moreover, we can write a generator of each principal ideal $Z_a^{(l_1, \dots, l_n)}$, $Z_{r,b}^{(l_1, \dots, l_n)}$ and $Z_{Br}^{(l_1, \dots, l_n)}$ as a product of cyclotomic polynomials as follows.

Proposition 3.3. For $l_1, \dots, l_n \geq 0$, the ideals $Z_a^{(l_1, \dots, l_n)}$, $Z_{r,b}^{(l_1, \dots, l_n)}$ and $Z_{Br}^{(l_1, \dots, l_n)}$ are principal. Moreover, we have

$$\begin{aligned}
Z_a^{(l_1, \dots, l_n)} &= \prod_{m \geq 1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m} \rfloor - \lfloor \frac{l_{\max}-1}{m} \rfloor - \lfloor \frac{1}{m} \rfloor} \mathbb{Z}[q, q^{-1}], \\
Z_{r,b}^{(l_1, \dots, l_n)} &= \prod_{1 \leq m \leq 2l_{\max}+1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m} \rfloor - \lfloor \frac{l_{\max}-1}{m} \rfloor - \lfloor \frac{1}{m} \rfloor + \sum_{1 \leq i \leq n, i \neq i_M} t_{i,m}} \mathbb{Z}[q, q^{-1}], \\
Z_{Br}^{(l_1, \dots, l_n)} &= \prod_{1 \leq m \leq 2l_{\max}+1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m} \rfloor - \lfloor \frac{l_{\max}-1}{m} \rfloor - \lfloor \frac{1}{m} \rfloor - \lfloor \frac{l_{\min}}{m} \rfloor + \sum_{1 \leq i \leq n, i \neq i_M, i_m} t_{i,m}} \mathbb{Z}[q, q^{-1}].
\end{aligned}$$

Proof. The assertion for $Z_a^{(l_1, \dots, l_n)}$ follows from

$$\{l\}_{q,i} = \prod_{m \geq 1} \Phi_m^{\lfloor \frac{l}{m} \rfloor - \lfloor \frac{l-i}{m} \rfloor}, \quad (5)$$

for $0 \leq i \leq l$. The assertion for $Z_{r,b}^{(l_1, \dots, l_n)}$ and $Z_{Br}^{(l_1, \dots, l_n)}$ follows from (5) and Theorem 3.1. \square

Corollary 3.4. For $l_1, \dots, l_n \geq 0$, we have

$$Z_a^{(l_1, \dots, l_n)} \cap Z_{Br}^{(l_1, \dots, l_n)} = \prod_{m \geq 1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m} \rfloor - \lfloor \frac{l_{\max}-1}{m} \rfloor - \lfloor \frac{1}{m} \rfloor + \max(0, \sum_{1 \leq i \leq n, i \neq i_M, i_m} t_{i,m} - \lfloor \frac{l_{\min}}{m} \rfloor)} \mathbb{Z}[q, q^{-1}].$$

Example 3.5. Let L be an n -component algebraically-split link with 0-framing. By Theorem 2.1 and Proposition 3.3, we have

$$\begin{aligned}
J_{L; \tilde{P}'_1, \dots, \tilde{P}'_1} &\in \Phi_1 \Phi_2 \Phi_3 \mathbb{Z}[q, q^{-1}], \\
J_{L; \tilde{P}'_2, \dots, \tilde{P}'_2} &\in \Phi_1^2 \Phi_2 \Phi_3 \Phi_4 \Phi_5 \mathbb{Z}[q, q^{-1}], \\
J_{L; \tilde{P}'_3, \dots, \tilde{P}'_3} &\in \Phi_1^3 \Phi_2^2 \Phi_3 \Phi_4 \Phi_5 \Phi_6 \Phi_7 \mathbb{Z}[q, q^{-1}].
\end{aligned}$$

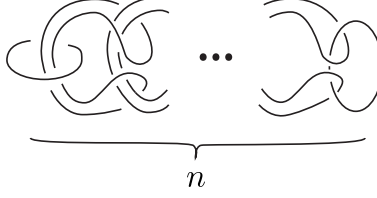


Figure 1: Milnor's link M_n

Let L be an n -component Brunnian link with $n \geq 3$. By Theorem 2.5 and Corollary 3.4, we have

$$\begin{aligned} J_{L; \tilde{P}'_1, \dots, \tilde{P}'_1} &\in \Phi_1^{n-2} \Phi_2 \Phi_3 \mathbb{Z}[q, q^{-1}], \\ J_{L; \tilde{P}'_2, \dots, \tilde{P}'_2} &\in \Phi_1^{2(n-2)} \Phi_2 \Phi_3 \Phi_4 \Phi_5 \mathbb{Z}[q, q^{-1}], \\ J_{L; \tilde{P}'_3, \dots, \tilde{P}'_3} &\in \Phi_1^{3(n-2)} \Phi_2^{n-1} \Phi_3 \Phi_4 \Phi_5 \Phi_6 \Phi_7 \mathbb{Z}[q, q^{-1}]. \end{aligned}$$

Let L be an n -component ribbon or boundary link with 0-framing. By Theorem 2.2 and Proposition 3.3, we have

$$\begin{aligned} J_{L; \tilde{P}'_1, \dots, \tilde{P}'_1} &\in \Phi_1^n \Phi_2 \Phi_3 \mathbb{Z}[q, q^{-1}], \\ J_{L; \tilde{P}'_2, \dots, \tilde{P}'_2} &\in \Phi_1^{2n} \Phi_2 \Phi_3 \Phi_4 \Phi_5 \mathbb{Z}[q, q^{-1}], \\ J_{L; \tilde{P}'_3, \dots, \tilde{P}'_3} &\in \Phi_1^{3n} \Phi_2^{n+1} \Phi_3 \Phi_4 \Phi_5 \Phi_6 \Phi_7 \mathbb{Z}[q, q^{-1}]. \end{aligned}$$

Example 3.6. For $n \geq 3$, let M_n be Milnor's n -component Brunnian link depicted in Figure 1. Note that M_3 is the Borromean rings. We have

$$J_{M_n; \tilde{P}'_1, \dots, \tilde{P}'_1} = (-1)^n q^{-2n+4} \Phi_1^{n-2} \Phi_2^{n-2} \Phi_3 \Phi_4^{n-3},$$

which we will prove in a forthcoming paper [12]. This implies that Theorem 2.4 is best possible for the divisibility by Φ_1 and Φ_3 of $J_{L; \tilde{P}'_1, \dots, \tilde{P}'_1}$ with L Brunnian. By Theorem 2.2, this also implies that each M_n is not ribbon or boundary.

4 Proof of Theorem 3.1

We prove Theorem 3.1.

For $a_1, \dots, a_m \in \mathbb{Z}[q, q^{-1}]$, let (a_1, \dots, a_m) denote the ideal in $\mathbb{Z}[q, q^{-1}]$ generated by $a_1, \dots, a_m \in \mathbb{Z}[q, q^{-1}]$.

For $l \geq 0$, recall that $I_l = (f_{l,0}, f_{l,1}, \dots, f_{l,l})$ with $f_{l,i} = \{l-i\}_q! \{i\}_q!$ for $0 \leq i \leq l$.

For $0 \leq k \leq l$, we have

$$(f_{l,0}, f_{l,1}, \dots, f_{l,k}) = \{l-k\}_q! (h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k})$$

with

$$\begin{aligned} h_{l,k,i} &= f_{l,i} / \{l-k\}_q! \\ &= \{l-i\}_{q,k-i} \{i\}_q!, \end{aligned}$$

for $1 \leq i \leq k$.

Set

$$\begin{aligned} I_{l,k} &= (h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k}), \\ g_{l,k} &= \text{GCD}(h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k}). \end{aligned}$$

Note that $I_{l,l} = I_l$ and $g_{l,l} = g_l$.

In what follows, for $a \in \mathbb{Z}[q, q^{-1}] \setminus \{0\}$ and $m \geq 1$, let $d_m(a)$ denote the largest integer i such that $a \in \Phi_m^i \mathbb{Z}[q, q^{-1}]$. For $0 \leq k \leq l$, we can write

$$g_{l,k} = \prod_{m \geq 1} \Phi_m^{d_m(g_{l,k})},$$

since each $h_{l,k,i}$ is a product of cyclotomic polynomials.

Lemma 4.1. *For $0 \leq k \leq l$, we have*

$$d_m(g_{l,k}) = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor & \text{for } 1 \leq m \leq k, \\ 0 & \text{for } k < m. \end{cases}$$

Proof. We have

$$\begin{aligned} d_m(g_{l,k}) &= \min\{d_m(h_{l,k,i}) \mid 0 \leq i \leq k\} \\ &= \min\{d_m(\{l-i\}_{q,k-i}\{i\}_q!) \mid 0 \leq i \leq k\} \\ &= \min\{\lfloor \frac{l-i}{m} \rfloor - \lfloor \frac{l-k}{m} \rfloor + \lfloor \frac{i}{m} \rfloor \mid 0 \leq i \leq k\} \\ &= \min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor \mid 0 \leq i \leq k\} - \lfloor \frac{l-k}{m} \rfloor. \end{aligned}$$

If $k < m$, then we have $d_m(g_{l,k}) = 0$ since $d_m(h_{l,k,k}) = d_m(\{k\}_q!) = 0$.

Let $1 \leq m \leq k$. Since we have

$$\lfloor \frac{l-(i+am)}{m} \rfloor + \lfloor \frac{i+am}{m} \rfloor = \lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor,$$

for $0 \leq i \leq k$ and $a \in \mathbb{Z}$, we have

$$\min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor \mid 0 \leq i \leq k\} = \min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor \mid 0 \leq i \leq m-1\}.$$

Here, for $0 \leq i \leq m-1$, we have $\lfloor \frac{i}{m} \rfloor = 0$ and $\lfloor \frac{l-i}{m} \rfloor$ takes the minimum with $i = m-1$. Thus we have

$$\begin{aligned} \min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor \mid 0 \leq i \leq m-1\} &= \lfloor \frac{l-(m-1)}{m} \rfloor \\ &= \lfloor \frac{l+1}{m} \rfloor - 1. \end{aligned}$$

This implies

$$d_m(g_{l,k}) = \lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor.$$

Hence we have the assertion. \square

Note that we have the latter part (4) of Theorem 3.1 as follows.

Corollary 4.2. *For $l \geq 0$, we have*

$$g_l = g_{l,l} = \prod_{m \geq 1} \Phi_m^{t_{l,m}}.$$

From now, we prove the following generalization of Theorem 3.1.

Proposition 4.3. *For $0 \leq k \leq l$, the ideal $I_{l,k}$ is the principal ideal generated by $g_{l,k}$.*

For $1 \leq k \leq l$, set

$$\tilde{g}_{l,k} = g_{l,k} / g_{l,k-1}.$$

We have

$$\begin{aligned} \tilde{g}_{l,k} &= \prod_{1 \leq m \leq k} \Phi_m^{\lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor - (\lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k+1}{m} \rfloor)} \\ &= \prod_{1 \leq m \leq k} \Phi_m^{\lfloor \frac{l-k+1}{m} \rfloor - \lfloor \frac{l-k}{m} \rfloor} \\ &= \prod_{\substack{m | l-k+1 \\ 1 \leq m \leq k}} \Phi_m. \end{aligned}$$

We use the following technical lemma.

Lemma 4.4. *For $1 \leq k \leq l$, we have*

$$(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = (\tilde{g}_{l,k}).$$

(Note that $g_{l,k-1} = \text{GCD}(\{l\}_{q,k-1}, \{l-1\}_{q,k-2}\{1\}_q, \dots, \{k-1\}_q!)$ divides $\{k-1\}_q!$.)

Proof of Proposition 4.3 by assuming Lemma 4.4. We use induction on k . For $k = 0$, clearly we have

$$I_{l,0} = (g_{l,0}) = (\{l\}_q!).$$

For $k \geq 1$, we have

$$\begin{aligned} I_{l,k} &= (h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k}) \\ &= (\{l\}_{q,k}, \{l-1\}_{q,k-1}\{1\}_q, \dots, \{l-k+1\}_q\{k-1\}_q!, \{k\}_q!) \\ &= (\{l-k+1\}_q(\{l\}_{q,k-1}, \{l-1\}_{q,k-2}\{1\}_q, \dots, \{k-1\}_q!), \{k\}_q!) \\ &= (\{l-k+1\}_q g_{l,k-1}, \{k\}_q!) \\ &= g_{l,k-1}(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) \\ &= (g_{l,k-1} \tilde{g}_{l,k}) \\ &= (g_{l,k}), \end{aligned}$$

where the second equality is given by

$$h_{l,k,0} = \{l - k + 1\}_q \cdot \{l - i\}_{q,k-i-1} \{i\}_q,$$

for $0 \leq i \leq k - 1$, and the third equality is given by the assumption of the induction.

Hence we have the assertion. \square

In what follows, we prove Lemma 4.4. We use the following two lemmas, which are well-known.

Lemma 4.5 (cf. Habiro [2, Lemma 4.1]). *For $a, b \geq 0$, the following conditions are equivalent.*

- (i) $(\Phi_a, \Phi_b) = \mathbb{Z}[q, q^{-1}]$
- (ii) $\frac{a}{b} \neq p^i$ for any prime number $p \geq 0$ and $i \in \mathbb{Z}$

Lemma 4.6. *Let $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{Z}[q, q^{-1}]$ such that $(a_i, b_j) = \mathbb{Z}[q, q^{-1}]$ for all $1 \leq i \leq m, 1 \leq j \leq n$. We have*

$$(a_1 a_2 \cdots a_m, b_1 b_2 \cdots b_n) = \mathbb{Z}[q, q^{-1}].$$

Proof of Lemma 4.4. It is enough to prove the following two equalities.

$$\text{GCD}(\{l - k + 1\}_q, \{k\}_q \frac{\{k - 1\}_q!}{g_{l,k-1}}) = \tilde{g}_{l,k}, \quad (6)$$

$$(\{l - k + 1\}_q / \tilde{g}_{l,k}, \{k\}_q \frac{\{k - 1\}_q!}{g_{l,k-1}} / \tilde{g}_{l,k}) = \mathbb{Z}[q, q^{-1}], \quad (7)$$

for $1 \leq k \leq l$.

First, we prove (6). Recall that

$$\tilde{g}_{l,k} = \prod_{\substack{m|l-k+1 \\ 1 \leq m \leq k}} \Phi_m. \quad (8)$$

Since $\{l - k + 1\}_q = \prod_{m|l-k+1} \Phi_m$ and $d_m(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 0$ for $m > k$, it is enough to check

$$d_m(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) \geq 1,$$

for $m|l - k + 1, 1 \leq m \leq k$. Indeed, we have

$$d_m(\{k\}_q) = \begin{cases} 1 & \text{for } m|k, \\ 0 & \text{for } m \nmid k, \end{cases} \quad (9)$$

$$d_m(\frac{\{k-1\}_q!}{g_{l,k-1}}) = \begin{cases} 0 & \text{for } m|k, \\ 1 & \text{for } m \nmid k. \end{cases} \quad (10)$$

Here, (9) is clear and (10) follows from

$$\begin{aligned}
d_m\left(\frac{\{k-1\}_q!}{g_{l,k-1}}\right) &= \left\lfloor \frac{k-1}{m} \right\rfloor - t_{l,k-1,m} \\
&= \left\lfloor \frac{k-1}{m} \right\rfloor - \left\lfloor \frac{l+1}{m} \right\rfloor + 1 + \left\lfloor \frac{l-k+1}{m} \right\rfloor \\
&= \left\lfloor \frac{pm+r-1}{m} \right\rfloor - \left\lfloor \frac{(p+p')m+r}{m} \right\rfloor + 1 + \left\lfloor \frac{p'm}{m} \right\rfloor \\
&= \left\lfloor \frac{r-1}{m} \right\rfloor + 1 \\
&= \begin{cases} 0 & \text{for } r = 0, \\ 1 & \text{for } 1 \leq r \leq m-1, \end{cases}
\end{aligned}$$

where, we write $k = mp + r$ and $l - k + 1 = mp'$ with $p, p' \geq 1$ and $0 \leq r \leq m-1$.

We prove (7). By Lemmas 4.5 and 4.6, it is enough to prove that there are no pair of integers $m, n \geq 1$ such that

- $\frac{n}{m} = p^i$ for a prime p and $i \in \mathbb{Z}$,
- $d_m(\{l-k+1\}_q/\tilde{g}_{l,k}) \geq 1$, and
- $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}) \geq 1$.

Note that

$$\{l-k+1\}_q/\tilde{g}_{l,k} = \prod_{\substack{m|l-k+1 \\ m>k}} \Phi_m.$$

Let $m|l-k+1$, $m > k$. Recall that for $n > k$, we have $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 0$. Assume that $1 \leq n \leq k$ and $n|m$, which implies $n|l-k+1$. The conditions $1 \leq n \leq k$ and $n|l-k+1$ imply $d_n(\tilde{g}_{l,k-1}) = 1$ by (8). By (9) and (10), we have $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 1$. Thus we have $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}) = 0$, which completes the proof. \square

Acknowledgments. This work was partially supported by JSPS Research Fellowships for Young Scientists. The author is deeply grateful to Professor Kazuo Habiro and Professor Tomotada Ohtsuki for helpful advice and encouragement.

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